Nonlinear and Buckling Analysis of Continuous Bars Lying on Rigid Supports

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Based on the theory of elastica, a parametric solution for the problem of nonlinear and buckling analysis of continuous bars on rigid supports is presented. Through the derived closed-form solutions of the equilibrium differential equations for each span, a nonlinear (transcendental) system of $3(q-1)$ equations with $4(q-1)$ unknowns was formulated. This system was further enriched by $(q-2)$ additional three-moment equations based on convenient compatibility conditions. The solution methodology was achieved by selecting values for the slope of the deflection of the first support, as well as for the elliptic integral appearing in the previous solutions (for the first member). Applications of the proposed methodology to continuous bars on three rigid supports and several numerical examples are given. The method is convenient for application to aerospace structural problems.

Introduction

THE second-order theory of linear buckling for continuous beams subjected to longitudinal compression forces has received considerable attention in recent years. Thorough examinations of this problem were made by Love[1] in 1944 and Timoshenko[2] in 1961. Both researchers considered a bar lying on rigid or elastic supports. The calculations of the critical compressive force were based on a three-moment equation that had three internal bending moments of the corresponding three consecutive supports.

On the other hand, the postbuckling behavior of an elastic column subjected to end compressive forces dates back to Euler.[1,2] However, for the problem of elastica of cantilevers, Love[1] and Timoshenko[2] underlined that the deflection after deformation can be calculated by first selecting a value of the angle of rotation of the free end. Also, in Ref. 3, a parametric solution of the problem of elastica of a simply supported bar subjected to a compressive force and a bending moment is given in terms of tabulated elliptic integrals of the first and second kind. Finally, the closed-form solutions of the strongly nonlinear differential equations analyzing the nonlinear and buckling of cantilevers subjected to a general terminal coplanar loading have been given in Refs. 4 and 5, taking into account the influence of transverse shear deformation.

In the present investigation, the nonlinear and buckling analysis of an elastic continuous bar on q rigid supports subjected to end compressive forces is presented. The methodology developed is based on the third-order theory, i.e., on the exact differential equation of the deflection curve. First, two arbitrary consecutive spans of the bar are analyzed in equivalent members, each of which can be considered to be a hinged bar with the one end moving and subjected to terminal compressive forces and indeterminate bending moments. Assuming that the compressive forces and flexural rigidities may vary from one span to the next, the strongly nonlinear equilibrium differential equation for each deformed member is formulated and the closed-form solution of this equation (expressed by elliptic integrals of the first and second kind) is derived. In the sequel, three transcendental equations are formulated for each span including the compressive forces, the indeterminate bending moments, and the coordinate distances of the moving supports after deformation. Moreover, using convenient compatibility conditions, a nonlinear equation relating the three internal bending moments of two consecutive spans is formulated. This equation is analogous to the three-moment equation for a continuous bar in the linear theory.[1,2] Consequently, for all spans of the bar, nonlinear (transcendental) system of $(4q-5)$ equations with $4(q-1)$ unknowns is constructed. A parametric (closed-form) solution of the previous system is achieved by selecting values for the slope of the deflection of the first support, as well as for the elliptic integral (for the first span) appearing in the derived equations.

The solution methodology that must be followed is illustrated in the applications examined in this paper. These applications concern the buckling of a continuous bar on three rigid supports. The bar is compressed by two equal terminal forces, while the ratio of the span lengths is considered equal to 2. In this case, the resulting transcendental system, expressed by seven equations, is solved in a closed form so that the slope of the deflected elastica is completely determined. Furthermore, the special case of a symmetric bar on three rigid supports is examined and several numerical results for both previous cases are presented, some of which are in agreement with those given by Timoshenko[2] for the simple problem of elastica in cantilevers.

The methodology proposed herein is convenient to apply to aerospace structural problems, as well as to engineering structures where large elastic deformations are required.

Analysis

Consider a slender continuous straight bar on rigid supports subjected to terminal axial compressive forces. Let $1,2,\ldots,q$ denote the consecutive supports; $M_2,M_3,\ldots,M_{q-1}$ the corresponding statically indeterminate bending moments; $\ell_1,\ell_2,\ldots,\ell_{q-1}$ the span lengths; and $\theta_1,\theta_2,\ldots,\theta_{q-1}$ the corresponding slopes of the deflection curve for each span. $\theta_{i,j}$ $(i,j=1,2,\ldots,q)$ denotes the angles of rotation of the right or left support i of the j span and $Q_{i,j}$ the corresponding support reactions. It is further assumed that the compressive force P and the flexural rigidity EJ may vary from one span to the next, but within each span these quantities are taken as constant.

In calculating the critical buckling loads for the previous bar, based on the theory of elastica, we must, in principle,

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analyze any two consecutive spans (ℓ_{n-1},ℓ_n), shown in Fig. 1, to the equivalent members ℓ_{n-1} and ℓ_n. Each of these members can be considered to be a hinged bar having the one movable end and loaded by terminal moments and axial and shear forces.

Based on the analysis given in Refs. 3 and 5, as well as on Fig. 1, and using the boundary conditions

$$\text{For } \theta_{n-1}=\theta_{n-1,n-1}\Rightarrow\frac{d\theta_{n-1}}{ds_{n-1}}=\frac{-M_{n-1}}{E_{n-1}J_{n-1}} \tag{1a}$$

$$\text{For } \theta_n=\theta_{n,n}\Rightarrow\frac{d\theta_n}{ds_n}=\frac{-M_n}{E_nJ_n} \tag{1b}$$

the exact nonlinear differential equations expressing the elastic curve of each span ℓ_{n-1} and ℓ_n take the following form, respectively:

$$\frac{d\theta_{n-1}}{k_{n-1,1}\sqrt{2}[A_{n-1}(\theta_{n-1})]^{1/2}}=-ds_{n-1} \tag{2a}$$

$$\frac{d\theta_n}{k_{n,1}\sqrt{2}[A_n(\theta_n)]^{1/2}}=-ds_n \tag{2b}$$

where

$$[A_{n-1}(\theta_{n-1})]^{1/2}=[k_{n-1,2}\sin\theta_{n-1,n-1}-\cos\theta_{n-1,n-1}$$
$$+k^2_{n-1,1}\lambda^2_{n-1}-k_{n-1,2}\sin\theta_{n-1}+\cos\theta_{n-1}]^{1/2} \tag{3a}$$

$$[A_n(\theta_n)]^{1/2}=[k_{n,2}\sin\theta_{n,n}-\cos\theta_{n,n}+k^2_{n,1}\lambda^2_n$$
$$-k_{n,2}\sin\theta_n+\cos\theta_n]^{1/2} \tag{3b}$$

in which

$$\lambda^2_{n-1}=(M_{n-1}/P_{n-1})^2/2,\quad \lambda^2_n=(M_n/P_n)^2/2$$

In these relations s denotes the arc length of the span after deformation, while $k_{\cdot,i}$ $(i=1,2)$ are coefficients given by

$$k^2_{n-1,1}=\frac{P_{n-1}}{E_{n-1}J_{n-1}},\qquad k_{n-1,2}=\frac{Q_{n-1,n-1}}{P_{n-1}}$$

$$k^2_{n,1}=\frac{P_n}{E_nJ_n},\qquad k_{n,2}=\frac{Q_{n,n}}{P_n} \tag{4}$$

Following the procedure given in Ref. 6 (p. 179), the integration of Eqs. (2) leads to

$$\mathfrak{F}_{n-1}=\frac{\mathfrak{F}(\gamma_{n-1},\mathscr{k}_{n-1})}{k_{n-1,1}(1+k^2_{n-1,2})^{1/4}}=-s_{n-1}+c_{n-1} \tag{5a}$$

$$\mathfrak{F}_n=\frac{\mathfrak{F}(\gamma_n,\mathscr{k}_n)}{k_{n,1}(1+k^2_{n,2})^{1/4}}=-s_n+c_n \tag{5b}$$

in which $\mathfrak{F}(\cdot,\cdot)$ is the incomplete elliptic integral of the first kind, while c_{n-1} and c_n are arbitrary constants of integration. The angles γ_{n-1} and γ_n and the modulii $\mathscr{k}_{n-1}$ and $\mathscr{k}_n$ of the previous elliptic integrals are given by the following relations:

$$\sin^2\gamma_{n-1}=[(1+k^2_{n-1,2})^{1/2}+k_{n-1,2}\sin\theta_{n-1}-\cos\theta_{n-1}]$$
$$\div[(1+k^2_{n-1,2})^{1/2}+k_{n-1,2}\sin\theta_{n-1,n-1}-\cos\theta_{n-1,n-1}$$
$$+k^2_{n-1,1}\lambda^2_{n-1}] \tag{6a}$$

$$\sin^2\gamma_n=\frac{(1+k^2_{n,2})^{1/2}+k_{n,2}\sin\theta_n-\cos\theta_n}{(1+k^2_{n,2})^{1/2}+k_{n,2}\sin\theta_{n,n}-\cos\theta_{n,n}+k^2_{n,1}\lambda^2_n} \tag{6b}$$

and

$$\mathscr{k}^2_{n-1}=[(1+k^2_{n-1,2})^{1/2}+k_{n-1,2}\sin\theta_{n-1,n-1}-\cos\theta_{n-1,n-1}$$
$$+k^2_{n-1,1}\lambda^2_{n-1}]/2(1+k^2_{n-1,2})^{1/2} \tag{7a}$$

$$\mathscr{k}^2_n=\frac{(1+k^2_{n,2})^{1/2}+k_{n,2}\sin\theta_{n,n}-\cos\theta_{n,n}+k^2_{n,1}\lambda^2_n}{2(1+k^2_{n,2})^{1/2}} \tag{7b}$$

By now, combining the boundary conditions

$$\text{For } \theta_{n-1}=\theta_{n-1,n-1}\Rightarrow s_{n-1}=0$$
$$\text{and for } \theta_{n-1}=\theta_{n,n-1}\Rightarrow s_{n-1}=\ell_{n-1} \tag{8a}$$

$$\text{For } \theta_n=\theta_{n,n}\Rightarrow s_n=0 \text{ and for } \theta_n=\theta_{n+1,n}\Rightarrow s_n=\ell_n \tag{8b}$$

together with Eqs. (5), we find

$$\ell_{n-1}=\frac{\tilde{\mathfrak{F}}_{n-1}-\bar{\mathfrak{F}}_{n-1}}{k_{n-1,1}(1+k^2_{n-1,2})^{1/4}} \tag{9}$$

$$\ell_n=\frac{\tilde{\mathfrak{F}}_n-\bar{\mathfrak{F}}_n}{k_{n,1}(1+k^2_{n,2})^{1/4}} \tag{10}$$

In the last equations, curled overbars are introduced to characterize quantities for $\theta_{n-1}=\theta_{n-1,n-1}$ and $\theta_n=\theta_{n,n}$, while straight overbars characterize quantities for $\theta_{n-1}=\theta_{n,n-1}$ and $\theta_n=\theta_{n+1,n}$, respectively. It must be underlined that the second member of Eq. (9) is a function of P_{n-1}, $\theta_{n-1,n-1}$, M_{n-1}, $Q_{n-1,n-1}$, M_n, and $\theta_{n,n-1}$, while the second member of Eq. (10) is a function of P_n, M_n, $Q_{n,n}$, $\theta_{n,n}$, $\theta_{n+1,n}$, and M_{n+1}.

The deflection equations, which must be solved to determine the shape of each buckled bar, are

$$dy=\frac{\sin\theta d\theta}{(d\theta/ds)};\quad dx=\frac{\cos\theta d\theta}{(d\theta/ds)} \tag{11}$$

where $d\theta/ds$ is given for each span by Eqs. (2a) and (2b), respectively. Using these well-known relations, as well as the results given in Ref. 6 (pp. 179-180), and introducing the boundary conditions

$$\text{For } \theta_{n-1}=\theta_{n-1,n-1}\Rightarrow y_{n-1}=x_{n-1}=0 \tag{12a}$$

$$\text{For } \theta_n=\theta_{n,n}\Rightarrow y_n=x_n=0 \tag{12b}$$

after some algebra, we find the following solutions for each span:

$$y_{n-1}(\theta_{n-1})=\frac{\sqrt{2}(k_{n-1,1}\lambda_{n-1}-A^{1/2}_{n-1})}{k_{n-1,1}(1+k^2_{n-1,2})}$$
$$-\frac{k_{n-1,2}(\tilde{\mathfrak{G}}_{n-1}-\mathfrak{G}_{n-1})}{k_{n-1,1}(1+k^2_{n-1,2})^{3/4}} \tag{13a}$$

$$x_{n-1}(\theta_{n-1})=\frac{\sqrt{2k_{n-1,2}}(k_{n-1,1}\lambda_{n-1}-A^{1/2}_{n-1})}{k_{n-1,1}(1+k^2_{n-1,2})}$$
$$+\frac{\tilde{\mathfrak{G}}_{n-1}-\mathfrak{G}_{n-1}}{k_{n-1,1}(1+k^2_{n-1,2})^{3/4}} \tag{13b}$$

and

$$\mathscr{y}_n(\theta_n) = \frac{\sqrt{2}(k_{n,1}\lambda_n - A_n^{1/2})}{k_{n,1}(1+k_{n,2}^2)} - \frac{k_{n,2}(\tilde{\mathcal{G}}_n - \mathcal{G}_n)}{k_{n,1}(1+k_{n,2}^2)^{1/4}} \tag{14a}$$

$$\mathscr{x}_n(\theta_n) = \frac{\sqrt{2}k_{n,2}(k_{n,1}\lambda_n - A_n^{1/2})}{k_{n,1}(1+k_{n,2}^2)} + \frac{\tilde{\mathcal{G}}_n - \mathcal{G}_n}{k_{n,1}(1+k_{n,2}^2)^{1/4}} \tag{14b}$$

in which

$$\mathcal{G}_{n-1} = \mathcal{G}(\gamma_{n-1}, \mathscr{k}_{n-1}) = 2\mathcal{E}(\gamma_{n-1}, \mathscr{k}_{n-1}) - \mathcal{F}(\gamma_{n-1}, \mathscr{k}_{n-1}) \tag{15a}$$

$$\mathcal{G}_n = \mathcal{G}(\gamma_n, \mathscr{k}_n) = 2\mathcal{E}(\gamma_n, \mathscr{k}_n) - \mathcal{F}(\gamma_n, \mathscr{k}_n) \tag{15b}$$

In these relations $\mathcal{E}(\cdot,\cdot)$ denotes the incomplete elliptic integral of the second kind. The deflections of the n and $(n+1)$ supports are obtained from Eqs. (13a) and (14a) by setting $\theta_{n-1} = \theta_{n,n-1}$ and $\theta_n = \theta_{n+1,n}$, respectively. The values of the elliptic integrals $\mathcal{G}_{n-1}$ and $\mathcal{G}_n$, which appear in the previous equations, can be determined by the conditions that the $\mathscr{y}_{n-1}(\theta_{n,n-1})$ and $\mathscr{y}_n(\theta_{n+1,n})$ deflections become equal to zero. In fact, using the well-known formula between the curvature function and the bending moment, Eqs. (2) give

$$A_{n-1}^{1/2}(\theta_{n,n-1}) = -\frac{1}{k_{n-1,1}\sqrt{2}} \frac{d\theta_{n-1}}{d\mathscr{s}_{n-1}}\bigg|_{\theta_{n-1}=\theta_{n,n-1}} = \frac{1}{k_{n-1,1}\sqrt{2}} \frac{M_n}{E_{n-1}J_{n-1}} \tag{16}$$

$$A_n^{1/2}(\theta_{n+1,n}) = -\frac{1}{k_{n,1}\sqrt{2}} \frac{d\theta_n}{d\mathscr{s}_n}\bigg|_{\theta_n=\theta_{n+1,n}} = \frac{1}{k_{n,1}\sqrt{2}} \frac{M_{n+1}}{E_n J_n} \tag{17}$$

Furthermore, inserting in these equations the quantities $k_{n-1,1}$ and $k_{n,1}$ given in Eqs. (4), we find that

$$\tilde{\mathcal{G}}_{n-1} - \bar{\mathcal{G}}_{n-1} = \frac{1}{k_{n-1,2}(1+k_{n-1,2}^2)^{1/4}} \frac{M_{n-1} - M_n}{(E_{n-1}J_{n-1}P_{n-1})^{1/2}} \tag{18}$$

$$\tilde{\mathcal{G}}_n - \mathcal{G}_n = \frac{1}{k_{n,2}(1+k_{n,2}^2)} \frac{M_n - M_{n+1}}{(E_n J_n P_n)^{1/2}} \tag{19}$$

On the other hand, the length ratios $\mathscr{x}_{\alpha,n-1}/\ell_{n-1}$ and $\mathscr{x}_{\alpha,n}/\ell_n$ ($\mathscr{x}_\alpha$ represents the coordinate distance of the moving support in Fig. 1) are obtained as follows. First, Eqs. (13b) and (14b) are formulated by setting $\theta_{n-1} = \theta_{n,n-1}$ and $\theta_n = \theta_{n+1,n}$, respectively; substitution of the elliptic integrals $\mathcal{G}_{n-1}$ and $\mathcal{G}_n$ given in Eqs. (15) into the results of these formulations gives

$$\mathscr{x}_{\alpha,n-1}k_{n-1,1}k_{n-1,2} = \frac{M_{n-1} - M_n}{(E_{n-1}J_{n-1}P_{n-1})^{1/2}} \tag{20}$$

$$\mathscr{x}_{\alpha,n}k_{n,1}k_{n,2} = \frac{M_n - M_{n+1}}{(E_n J_n P_n)^{1/2}} \tag{21}$$

It is obvious that, for each span of the bar, a set of three independent nonlinear (transcendental) equations is formulated to connect the two indeterminate bending moments of the end supports [i.e., for the $(n-1)$ span, these equations are expressed by Eqs. (9), (18), and (20)]. So, for all of the structure, a set of $3(q-1)$ nonlinear equations exists that relate the $(q-2)$ indeterminate bending moments of the $(q-2)$ consecutive indeterminate supports. The previous set of $3(q-1)$ equations includes $4(q-1)$ unknown functions, namely, the $(q-1)$ compressive forces, $(q-2)$ indeterminate bending moments, q slopes of the deflected elastica, and $(q-1)$ coordinates $\mathscr{x}_\alpha$ after deformation [or the $(q-1)$ support reactions]. Consequently, for the calculation of all previous unknowns, a significant number of new equations must be formulated.

The Three-Moment Bending Equation

For the two consecutive spans shown in Fig. 1, we introduce the following compatibility conditions:

$$d\theta_n/d\mathscr{s}_n|_{\theta_n=\theta_{n+1,n}} = \frac{-M_{n+1}}{E_n J_n} \tag{22}$$

$$\frac{\kappa_n(\theta_{n,n})}{\kappa_{n-1}(\theta_{n,n-1})} = \frac{E_{n-1}J_{n-1}}{E_n J_n} = \omega \tag{23}$$

$$\theta_{n,n-1} = -\theta_{n,n} \tag{24}$$

in which κ denotes the curvature function.

Equation (22) together with Eq. (3b) gives

$$k_{n,2}\sin\theta_{n,n} - \cos\theta_{n,n} - k_{n,2}\sin\theta_{n+1,n} + \cos\theta_{n+1,n} = \frac{1}{2k_{n,1}^2}\frac{M_{n+1}^2}{E_n^2 J_n^2} - k_{n,1}^2\lambda_n^2 \tag{25}$$

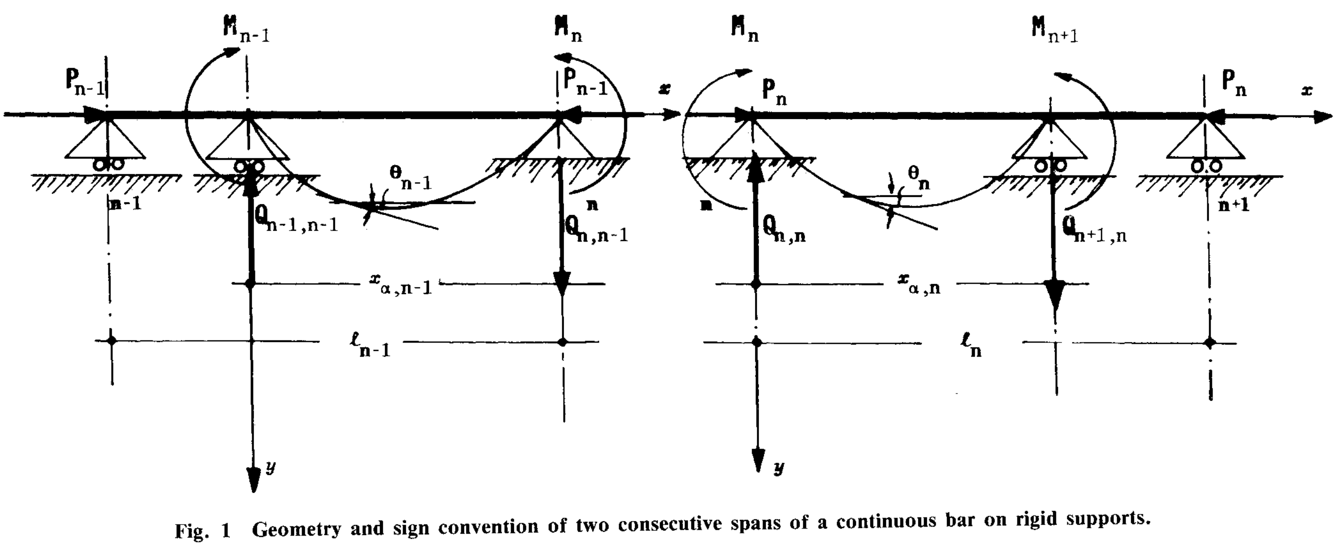

Fig. 1 Geometry and sign convention of two consecutive spans of a continuous bar on rigid supports.

Also, combining Eq. (23) with Eqs. (3) and (24), after some algebra, leads to

$$k_{n-1,2}\sin\theta_{n-1,n-1}-\cos\theta_{n-1,n-1}+k_{n-1,2}\sin\theta_{n,n}$$
$$+\cos\theta_{n,n}=\frac{k_{n,1}^4\lambda_n^2}{\omega^2 k_{n-1,1}^2}-k_{n-1,1}^2\lambda_{n-1}^2 \tag{26}$$

Adding by parts Eqs. (25) and (26), we find that

$$(k_{n-1,2}\sin\theta_{n-1,n-1}-\cos\theta_{n-1,n-1})$$
$$+(k_{n-1,2}+k_{n,2})\sin\theta_{n,n}-(k_{n,2}\sin\theta_{n+1,n}-\cos\theta_{n+1,n})$$
$$=\frac{k_{n,1}^4\lambda_n^2}{\omega^2 k_{n-1,1}^2}-k_{n-1,1}^2\lambda_{n-1}^2-k_{n,1}^2\lambda_n^2+\frac{1}{2k_{n,1}^2}\frac{M_{n+1}^2}{E_n^2 J_n^2} \tag{27}$$

Equation (27) is transformed via the following procedure. From Eq. (7a), obtain

$$k_{n-1,2}\sin\theta_{n-1,n-1}-\cos\theta_{n-1,n-1}$$
$$=(2\ell_{n-1}^2-1)(1+k_{n-1,2}^2)^{1/2}-k_{n-1,1}^2\lambda_{n-1}^2 \tag{28}$$

and from Eq. (6a) for $\theta_{n-1}=\theta_{n,n-1}$ and Eq. (24)

$$(k_{n-1,2}+k_{n,2})\sin\theta_{n,n}=-(1+k_{n-1,2}^2)^{1/2}(2\ell_{n-1}^2\sin^2\bar{\gamma}_{n-1}-1)$$
$$-(2\ell_n^2-1)(1+k_{n,2}^2)^{1/2}+k_{n,1}^2\lambda_n^2 \tag{29}$$

Finally, combining Eq. (6b) for $\theta_n=\theta_{n+1,n}$ with Eq. (7b), we find

$$k_{n,2}\sin\theta_{n+1,n}-\cos\theta_{n+1,n}=(1+k_{n,2}^2)^{1/2}(2\ell_n^2\sin^2\bar{\gamma}_n-1) \tag{30}$$

Now, introducing Eqs. (28-30) into Eq. (27), the following equation is derived:

$$(1+k_{n-1,2}^2)^{1/2}\ell_{n-1}^2\cos^2\bar{\gamma}_{n-1}+(1+k_{n,2}^2)^{1/2}\ell_n^2\cos^2\bar{\gamma}_n$$
$$=\frac{1}{4}\left(\frac{M_n^2}{E_{n-1}J_{n-1}P_{n-1}}+\frac{M_{n+1}^2}{E_nJ_nP_n}\right) \tag{31}$$

which, based on Eqs. (6) and (7), takes the following final form:

$$\frac{M_{n-1}-M_n}{x_{\alpha,n-1}P_{n-1}}(\sin\theta_{n-1,n-1}-\sin\theta_{n,n-1})+\frac{M_n-M_{n+1}}{x_{\alpha,n}P_n}$$
$$\times(\sin\theta_{n,n}-\sin\theta_{n+1,n})=(\cos\theta_{n-1,n-1}-\cos\theta_{n+1,n})$$
$$+\frac{1}{2}\left(\frac{M_n^2-M_{n-1}^2}{E_{n-1}J_{n-1}P_{n-1}}+\frac{M_{n+1}^2-M_n^2}{E_nJ_nP_n}\right) \tag{32}$$

Equation (32) [or (31)] is the three-bending moment equation for a continuous straight buckled bar, based on the third-order theory (the theory of elastica). Writing this equation for each intermediate support of the bar and including the possible loading of the first and last supports, we obtain a number of $(q-2)$ equations relating all of the unknown bending moments with the axial forces, the support reactions (or the coordinates distances x_α), and the slopes of the deflection curve.

For the direct use of the present solutions, one may choose a value for the angle $\theta_{1,1}$ of the first support and for the elliptic integral $\mathfrak{F}_1(\varphi_1,\ell_1)=\mathfrak{F}_1(\pi/2,\ell_1)-\mathfrak{F}_1(\varphi_1,\ell_1)$, so that a parametric solution of the problem can be achieved. Instead of the selection of these more abstract values $(\theta_{1,1},\mathfrak{F}_1)$, one may prefer to choose values of M_2 and $x_{\alpha,1}$ or P, as indicated in Ref. 3. Here, values of the quantities $\theta_{1,1}$ and $\mathfrak{F}_1$ were given in order to be more relative to the solution already developed by Love[1] and Timoshenko[2] for the simple problem of cantilevers.

Applications and Numerical Results

As a first application of the proposed methodology, we shall examine a practical problem usually appearing in engineering structures made of high-strength alloy steels, such as bridges, frames, ships, aircraft, etc., where there are a variety of stability problems in modern design. Consider a slender continuous bar lying on three rigid supports (1,2,3), from which the intermediate one is a hinge and the extremes are simple supports. This bar is compressed by forces P applied at the ends. The lengths of the two spans are ℓ_1 and ℓ_2, respectively, (with $\ell_2=2\ell_1$) and the flexural rigidities are considered equal to EJ. For this case, the Euler critical buckling load (based on the second-order theory) is given by (see Ref. 2, p. 68),

$$P_{\mathrm{cr},E}=3.72EJ/\ell_1^2=14.90EJ/\ell_2^2 \tag{33}$$

For the first span of the bar, the combination of Eqs. (9) and (33) leads to

$$P/P_{\mathrm{cr},E}=\mathfrak{F}_1^2/3.72(1+k_{1,2}^2)^{1/2} \tag{34}$$

while the combination of Eqs. (18), (20), and (9) gives

$$x_{\alpha,1}/\ell_1=(1+k_{1,2}^2)^{1/2}\mathfrak{G}_1/\mathfrak{F}_1 \tag{35}$$

in which, because of Eq. (6a) $[k_{1,1}^2\lambda_1^2=(M_1/P)^2/2=0]$, we have

$$\mathfrak{F}_1=\mathfrak{F}_1(\varphi_1,\ell_1)=\mathfrak{F}_1(\pi/2,\ell_1)-\mathfrak{F}_1(\bar{\gamma}_1,\ell_1) \tag{36a}$$

$$\mathfrak{G}_1=\mathfrak{G}_1(\varphi_1,\ell_1)=\mathfrak{G}_1(\pi/2,\ell_1)-\mathfrak{G}_1(\bar{\gamma}_1,\ell_1) \tag{36b}$$

In these relations, the angle φ_1 of the elliptic integrals is given by (see Ref. 6, p. 13),

$$\varphi_1=\cos^{-1}\left(\frac{\sin\bar{\gamma}[(1-\ell_1^2)(1-\ell_1^2\sin^2\bar{\gamma}_1)]^{1/2}}{1-\ell_1^2\sin^2\bar{\gamma}}\right) \tag{37}$$

Also, using the relation

$$M_2=x_{\alpha,1}Q_{1,1}$$

we derive

$$\frac{M_2}{\ell_1P_{\mathrm{cr},E}}=\frac{x_{\alpha,1}}{\ell_1}\frac{Q_{1,1}}{P_{\mathrm{cr},E}}=\frac{x_{\alpha,1}}{\ell_1}\frac{Q_{1,1}}{P}\left(\frac{P}{P_{\mathrm{cr},E}}\right)=\frac{k_{1,2}\mathfrak{F}_1\mathfrak{G}_1}{3.72} \tag{38}$$

On the other hand, the three-moment equation (32) takes the following form:

$$-k_{1,2}(\sin\theta_{1,1}-\sin\theta_{2,1})+k_{2,2}(-\sin\theta_{2,1}-\sin\theta_{3,2})$$
$$=\cos\theta_{1,1}-\cos\theta_{3,2} \tag{39}$$

By now, solving Eq. (7a) for $k_{1,2}$ we find

$$k_{1,2}=-\sin\theta_{1,1}\cos\theta_{1,1}\pm\{\sin^2\theta_{1,1}\cos^2\theta_{1,1}-[(2\ell_1^2-1)^2$$
$$-\sin^2\theta_{1,1}][(2\ell_1^2-1)^2-\cos^2\theta_{1,1}]\}^{1/2}$$
$$\div(2\ell_1^2-1)^2-\sin^2\theta_{1,1} \tag{40}$$

Choosing a value for the angle $\theta_{1,1}$ and various values for the modulus $\mathscr{k}_1$ and the angle φ_1 of the elliptic integral $\mathcal{F}_1(\varphi_1,\mathscr{k}_1)$, the quantity $k_{1,2}$ can be calculated from Eq. (40), as well as the angle $\bar{\gamma}_1$ from Eq. (37). Notice here that the selection of the values $\mathscr{k}_1$ and φ_1 must be such that

$$\mathscr{x}_{\alpha,1}/\ell_1 > 0 \Leftrightarrow \mathcal{G}_1 > 0 \Leftrightarrow 2\mathcal{E}_1 - \mathcal{F}_1 > 0 \tag{41}$$

Furthermore, Eqs. (34), (35), (38), and (6a) are used to evaluate the dimensionless ratios $P/P_{cr,E}$, $\mathscr{x}_{\alpha,1}/\ell_1$, and $M_2/\ell_1 P_{cr,E}$, as well as the angle $\theta_{2,1} = -\theta_{2,2}$.

In Table 1, the previous ratios and the angle $\theta_{2,1}$ are derived for $\theta_{1,1} = 30$ deg, $\varphi_1 = 120$ deg, and various values of modulus $\mathscr{k}_1$. In this case, for the validity of the inequality equation (41), the modulus $\mathscr{k}_1$ must be smaller than $\sin^{-1}$ 60 deg. Since the values P, M_2, and $\theta_{2,1}$ are now known based on Eq. (33), Eqs. (34), (35), and (38) for the second span can be written as

$$P/P_{cr,E} = \mathcal{F}_2^2/14.90(1 + k_{2,2}^2)^{1/2} \tag{42}$$

$$\mathscr{x}_{\alpha,2}/\ell_2 = (1 + k_{2,2}^2)^{1/2}\mathcal{G}_2/\mathcal{F}_2 \tag{43}$$

and

$$M_2/\ell_2 P_{cr,E} = k_{2,2}\mathcal{G}_2\mathcal{F}_2/14.90 \tag{44}$$

For the evaluation of the dimensionless ratio $\mathscr{x}_{\alpha,2}/\ell_2$, one may use Figs. 4 and 5 of Ref. 3, which have been conveniently prepared to permit the user to determine the values $\mathscr{x}_\alpha/\ell$, $P/P_{cr,E}$, and $M/\ell P_{cr,E}$ corresponding to particular values of the modulus and the angle of the elliptic integral $\mathcal{F}$ and vice versa. For this reason, Eqs. (42) and (43) can be written in a form analogous to Eqs. (10) and (20) of Ref. 3 as follows:

$$14.90P/\pi^2 P_{cr,E} = \mathcal{F}_2^2/\pi^2(1 + k_{2,2}^2)^{1/2} \tag{45}$$

$$14.90(M_2/\pi^2 \ell_1 P_{cr,E})(\ell_1/\ell_2) = k_{2,2}\mathcal{F}_2\mathcal{G}_2/\pi^2 \tag{46}$$

Fitting the values of columns 6 and 7 of Table 1 according to Eqs. (45) and (46) and using Fig. 5 of Ref. 5, we find the following solutions for the dimensionless ratio $\mathscr{x}_{\alpha,2}/\ell_2$:

$$\text{For } P/P_{cr,E} = 1.80 \text{ and } M_2/\ell_1 P_{cr,E} = -0.21 \Rightarrow \mathscr{x}_{\alpha,2}/\ell_2 = 0.22 \tag{47a}$$

$$\text{For } P/P_{cr,E} = 1.04 \text{ and } M_2/\ell_1 P_{cr,E} = +0.91 \Rightarrow \mathscr{x}_{\alpha,2}/\ell_2 = 0.25 \tag{47b}$$

Now, solving the nonlinear system of Eqs. (42-44) for $k_{2,2}$, $\mathcal{F}_2$, and $\mathcal{G}_2$, one may find for the dimensionless ratio $k_{2,2}$,

$$k_{2,2} = -0.41 \tag{48a}$$

$$k_{2,2} = +1.77 \tag{48b}$$

Furthermore, introducing the resulting value $k_{2,2} = -0.41$ (or $+1.77$) into the three-moment equation (39), we obtain

$$\theta_{3,2} = -71.50 \text{ deg} \tag{49a}$$

$$\theta_{3,2} = +55.00 \text{ deg} \tag{49b}$$

Consequently, for the problem being examined, the deflected elastica is completely determined and presented in Fig. 2.

As a second application, consider the case when the lengths of the two spans of a continuous bar on three sup-

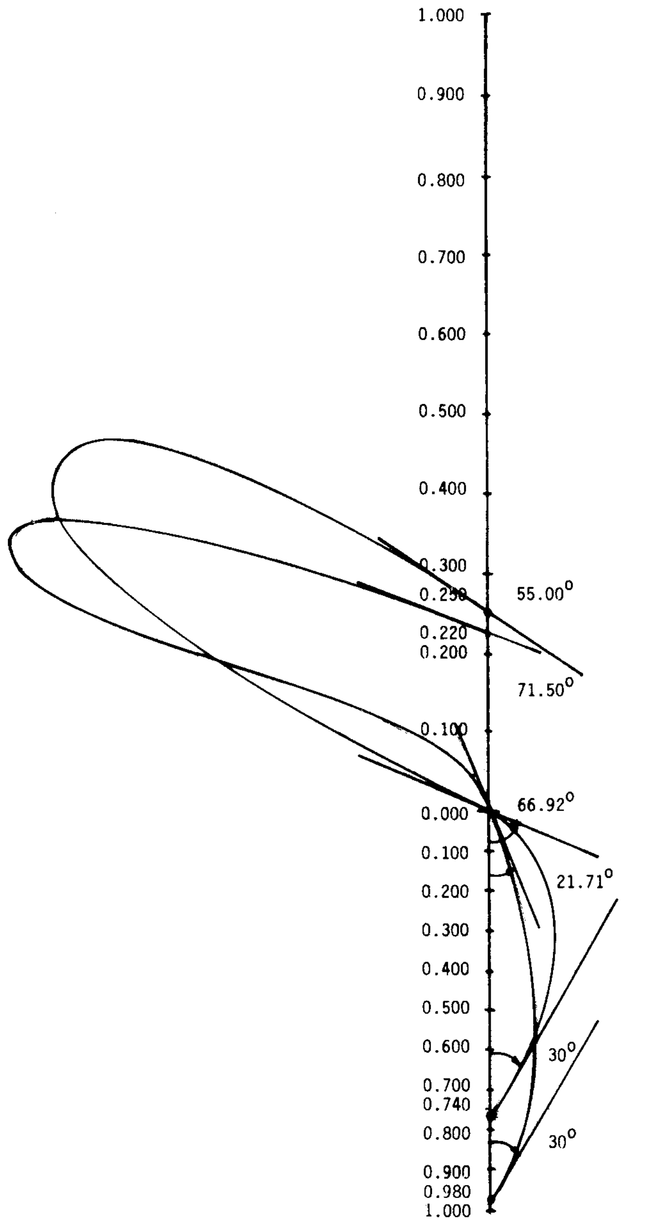

Fig. 2 Elastica shape resulting from a pure compressive load of a continuous bar on three supports.

Table 1 Load deflection data for a buckled continuous bar on three rigid supports for $\theta_{1,1} = 30$ deg, $\varphi_1 = 120$ deg, and for various values of the modulus $\mathscr{k}_1$

$\sin^{-1}\mathscr{k}_1$, deg	$\mathcal{F}_1(\varphi_1,\mathscr{k}_1)$	$\mathcal{G}_1(\varphi_1,\mathscr{k}_1)$	$k_{1,2}$	$\bar{\gamma}_1$, deg	$P/P_{cr,E}$	$M_2/\ell_1 P_{cr,E}$	$\mathscr{x}_{\alpha,1}/\ell_1$	$\theta_{2,1}$, deg	$\theta_{2,2}$, deg	$\mathscr{x}_{\alpha,2}/\ell_2$	$k_{2,2}$	$\theta_{3,2}$, deg
10	2.114	2.036	−0.180	30.380	1.180	−0.210	0.980	−21.710	+21.710	0.220	−0.410	−71.500
20	2.174	1.864	+0.170	31.560	1.250	+0.190	0.870	+19.900	−19.900	—	—	—
30	2.282	1.572	+0.580	33.560	1.210	+0.560	0.800	+65.500	−65.500	—	—	—
40	2.451	1.161	+1.190	37.00	1.040	+0.910	0.740	+66.920	−66.920	0.250	+1.770	+55.000
50	2.707	0.617	+1.190	49.080	1.270	+0.530	0.350	+85.560	−85.560	—	—	—

Table 2 Load deflection data for a buckled continuous symmetric bar on three rigid supports

	$\theta_{1,1}$, deg					
Ratio	0	20	40	60	80	100
$P/P_{cr,E}$	1	1.015	1.065	1.152	1.292	1.507
$x_{\alpha,1}/\ell$	1	0.970	0.878	0.747	0.563	0.358
y_{max}/ℓ	0	0.110	0.211	0.297	0.359	0.396

ports are equal to ℓ. Then, because of the symmetry of the structure, we have

$$\theta_{1,1}=\theta_{3,2}, \quad k_{2,1}=k_{2,2}=M_2/x_\alpha P \tag{50}$$

Thus, the three-moment equation (39) leads to

$$2k_{2,1}\sin\theta_{1,1}=0 \tag{51}$$

Because $\sin\theta_{1,1}\neq 0$, the result is that

$$k_{2,1}=k_{2,2}=0 \Leftrightarrow M_2=0 \tag{52}$$

This conclusion coincides with that given by Timoshenko[2] (pp. 67-68) for the same problem solved according to the linear buckling theory (second-order theory). Thus, each span can be considered as a bar with the one end hinged and the other simply supported. Consequently, for angle $\bar{\gamma}_1$ of the elliptic integral $\mathfrak{F}_1$, we have

$$\sin^2\bar{\gamma}_1=1$$

The first value of $\bar{\gamma}_1$, whch secures $k_{1,1}\ell_1\neq 0$ is

$$\bar{\gamma}_1=-\pi/2$$

Introducing the value $\bar{\gamma}_1$ into Eq. (9), we find

$$k_{1,1}\ell_1=2\mathfrak{F}_l(\pi/2,k_1) \tag{53}$$

in which

$$k_1^2=\sin^2\theta_{1,1}/2 \tag{54}$$

Squaring Eq. (53) results in

$$P_{cr}=4\mathfrak{F}_1^2(\pi/2,k_1)EJ/\ell_1^2 \tag{55}$$

It is obvious now that if

$$\theta_{1,1}=0$$

it may be derived from Eq. (54) that

$$k_1^2=0$$

Consequently, Eq. (55) can be written as

$$P_{cr}=\frac{4\mathfrak{F}_1^2(\pi/2,0)EJ}{\ell_1^2}=\frac{\pi^2 EJ}{\ell_1^2}=P_{cr,E} \tag{56}$$

i.e., the critical buckling load of a hinged bar is derived according to the second-order theory. As the value of the angle $\theta_{1,1}$ increases, the integral $\mathfrak{F}_1(\pi/2,k_1)$ and the load P_{cr} also increase. From Eqs. (13), after some algebra, we find

$$y_1(\theta_1)=\frac{-\sqrt{2}(\cos\theta_1-\cos\theta_{1,1})^{1/2}}{k_{1,1}} \tag{57a}$$

$$x_{\alpha,1}(\theta_1)=\frac{2[2\mathcal{E}_1(\pi/2,k_1)-\mathfrak{F}_1(\pi/2,k_1)]}{k_{1,1}} \tag{57b}$$

We note here that from Eq. (57a), one may conclude that the maximum deflection (y_{max}) occurs for the value $\cos\theta_1=1$, i.e.,

$$y_{max}=-\sqrt{2}(1-\cos\theta_{1,1})^{1/2}/k_{1,1} \tag{58}$$

In Table 2, the values of the dimensionless ratios $P_{cr}/P_{cr,E}$, $x_{\alpha,1}/\ell_1$, and y_{max}/ℓ_1 are given for various values of the angle $\theta_{1,1}$. We observe that, the two first ratios included in this table coincide with the corresponding ratios of Table 2-4 given by Timoshenko.[2] Also, the values of the third ratio in Table 2 are one-half of the corresponding values of the ratio given in Table 2-4 by Timoshenko.[2] These results are reasonable, because the values given in Ref. 2 relate to the elastica for a cantilever bar, whereas those here concern the elastica of a hinged bar.

Conclusions

This paper has given a parametric solution to the problem of nonlinear and buckling analysis of continuous bars on rigid supports based on the theory of elastica. First, using convenient boundary conditions, the closed-form solutions of the strongly nonlinear equilibrium differential equations of each deformed member were found (expressed by elliptic integrals of the first and second kind); these nonlinear (transcendental) equations were formulated for each span, including the compressive forces, terminal indeterminate bending moments, and coordinate distances of the moving supports after deformation. In the sequel, through convenient compatibility conditions, a nonlinear (transcendental) three-moment equation of two consecutive spans was formulated. Consequently, for all of the spans of the bar, a nonlinear system of $(4q-5)$ equations with $4(q-1)$ unknowns was constructed.

A solution methodology of this system was developed through an application to the nonlinear and buckling analysis of continuous bars on three rigid supports. Several numerical results were also given concerning the critical buckling load, indeterminate bending moments, coordinate distances after deformation, and exact form of the deflected elastica.

The proposed methodology is convenient for aerospace structural problems, as well as for engineering structures where large elastic deformations are encountered.

References

[1]Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, New York, 1944, Chaps. XVIII-XIX.

[2]Timoshenko, P. S. and Gere, J. M., *Theory of Elastic Stability*, McGraw-Hill Book Co., New York, 1961, pp. 76-78.

[3]Griner, G. M., "A Parametric Solution to the Elastic Pole-Vaulting Pole Problem," *Journal of Applied Mechanics, Transactions of ASME*, Vol. 51, 1984, pp. 409-414.

[4]Panayotounakos, D. E. and Theocaris, P. S., "Nonlinear Analysis of Cantilever Bars Due to Terminal Moments," *Ingenieur-Archiv*, Vol. 49, 1980, pp. 73-79.

[5]Theocaris, P. S. and Panayotounakos, D. E., "Exact Solution of the Non-Linear Differential Equation Concerning the Elastic Line of a Straight Bar Due to Terminal Loading," *International Journal of Non-Linear Mechanics*, Vol. 17, 1982, pp. 395-402.

[6]Byrd, P. F. and Friedman, M. D., *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer-Verlag, New York, 1971.